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MAGNETIC LOSS AND INSTABILITIES IN FERRITE GARNET TUNED RF CAVITIES FOR SYNCHROTRONS

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Abstract

This note is devoted to the behaviour of high power rf cavities with transversely biased ferrite garnet used for varying the frequency in rapid cycling synchrotrons of the type designed in TRIUMF and SSCL. The magnetic loss and nonlinear effect, though essential for the operation and determination of limits of the application, has not been previously studied to any great extent. A method of analysis using minimum specific details is developed and simple formulae, physics and estimates of the trend of magnetic loss, nonlinear frequency shift and possible instabilities in the cavities as a function of rf power level and ferrite garnet parameters are presented.

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1. Introduction

Tuning of rf cavities by transversely biased ferrite garnet is a relatively new method [1,2] used at present for the design of state-of-the-art tuned rf cavities for rapid cycling syncrotrons. The cavities of the considered type have a specific axial geometry. In particular, this is the design used at TRIUMF and SSCL [3,4]. Essential factors and limits of the application are determined by magnetic loss and associated nonlinear phenomena in the ferrite garnet at high rf power level. They have not been systematically studied. There are a lot of investigations on the behaviour of ferrite garnets at microwaves frequencies but at lower frequencies one encounters different phenomena.

In this note, a method of analysis within a general problem statement combined with estimates for the system in TRIUMF will be presented.

It seems reasonable to divide the processes into slow and fast. The magnetization of the ferrite garnet follows the magnetic rf field almost simultaneously and the nonlinearity of the magnetic permeability gives rise to relatively fast nonlinear processes as compared to thermal and thermoelastic processes. The latter can be regarded as slow and taken into account parametrically by considering the parameters of the fast system dynamics to be smoothly varying. The slow processes can be controlled to some extent by using feedback loops while the fast ones are more basic and they will be the focus of our investigation.

One of the questions under investigation, about the nonlinear shift of the resonance frequency of the rf cavity, was considered recently [5] but within a simplified model. The demagnetization field in the material was implicitly ignored. The relaxation of the ferrite magnetization was ignored as well. Both factors are of importance for linear and nonlinear behaviour of the considered systems.

2. Problem statement

We proceed from the well-known general equation governing behaviour of magnetization $\mathbf{M}(\mathbf{r},t)$ of magnetic material

$$\frac{\partial \mathbf{M}}{\partial t} = g\mathbf{H}_{ef} \times \mathbf{M} + \mathbf{R} \tag{1}$$

where g is the gyromagnetic ratio, $g \approx 2.8$ MHz/Oe for most materials, $\mathbf{R}$ describes dissipation processes and $\mathbf{H}_{ef}$ is the effective field determined as

$$\mathbf{H}_{ef}(\mathbf{r},t) = -\frac{\delta U}{\delta \mathbf{M}(\mathbf{r},t)} \tag{2}$$

U is the energy of the ferrite including its interaction with external field, demagnetization field, anisotropy and exchange.

The dissipation term $\mathbf{R}$ in (1) will be specified by the Landau-Lifschitz form

$$\mathbf{R} = \frac{\alpha g}{M}\mathbf{M} \times (\mathbf{H}_{ef} \times \mathbf{M}) \tag{3}$$

The magnitude $M = |\mathbf{M}|$, determined via (1), is not changed by this kind of damping. α is a dimensionless parameter depending on the ground state of material which in turn depends on the applied field.

Such a modelling is successfully used in micromagnetics to describe spin waves, domain wall motion and the uniform precession of $\mathbf{M}$ at microwaves and radio frequencies (e.g. [6,7]).

When ignoring inhomogeneities of the material and considering the applied fields exceeding the hysteresis area so that the domain walls in the material practically disappear, one may neglect the contribution to $\mathbf{H}_{ef}$ due to the energy of anisotropy and exchange and take

$$\mathbf{H}_{ef} = \mathbf{H} \tag{4}$$

where $\mathbf{H}$ is the magnetic field in the garnet.

These equations, together with Maxwell's equations and boundary conditions, form the basis of our analysis of the fast processes in the system.

The magnetic material in the rf cavities of interest is yttrium-iron garnet ring discs biased along the axis of the discs (taken for axis z) by external dc and ac magnetic field,

$$\mathbf{H}_e = (0, 0, H_e).$$

The amplitude H_e is relatively large, close to saturation, and exceeds considerably the rf field. Approximately

$$\mathbf{h}_e \perp \mathbf{H}_e$$

where $\mathbf{h}_e$ is the rf field in the cavity at the surface of the garnets in their vicinity. Both fields are assumed to be inhomogeneous but to vary smoothly at spatial scales compared with or smaller than the thickness of the garnet ring disks.

3. Linear and nonlinear permeability

Proceeding from (1) to (4) let us rewrite them in the form

$$\dot{\mathbf{M}} + \frac{\alpha}{M}\dot{\mathbf{M}} \times \mathbf{M} = g_{ef}\mathbf{H} \times \mathbf{M} \tag{5}$$

where $\dot{\mathbf{M}} = \partial \mathbf{M}/\partial t$ and $g_{ef} = (1 + \alpha^2)g$.

The low loss materials, with α much less than 1, are of interest so without introducing a large error we may replace g_{ef} in the right of (5) by g. Note that in contrast to what is written in some manuals, eq.(5) is exact and not an approximation of (1) to (4) valid only for small oscillations of $\mathbf{M}$ with respect to its unperturbed value $(0, 0, M)$.

Remark 1. In applications of (5) in microwaves, the parameter α is related to the line widths ΔH_k of spin-wave resonance and ΔH of uniform spin-precession resonance at microwave frequency ω_o as

$$\alpha = \frac{g\Delta H_k}{\omega_o}$$

and with ΔH instead of ΔH_k respectively. For the garnet used in the booster synchrotron in TRIUMF $\Delta H_k = 1.5$ Oe and $\Delta H = 40$ Oe at $\omega_o/2\pi = 9.4$ GHz, so

$$\alpha \approx 4.5 \cdot 10^{-4} \text{ and } 1.2 \cdot 10^{-2} \text{ respectively.}$$

What value of α is relevant for the case? We have not found any discussion of this matter. The question is of essential importance and let us dwell on it. The linewidth ΔH of the uniform ferromagnetic resonance is broadened considerably as compared to the linewidth of a single crystal as illustrated by Fig. 1. This is due to inhomogenities and polycrystaline structure of the material. The quantity ΔH_k, being assocciated with parametrical excitation of the spin-wave with the lowest threshold of the excitation, is closer to the linewidth of a single crystal (or a small subvolume of the material) than to ΔH. Since the radio frequencies of interest are much below the resonance microwave frequencies, all the partial contributions to the uniform mode resonance curve from such subvolumes decay at $\omega \ll \omega_o$, approximately the same as a single crystal decay. Therefore it seems reasonable for the application of interest to take for α the quantity associated with ΔH_k rather than with ΔH.

Such an estimate of α, via ΔH_k, we find in Rodrigue [8] (but without an explanation). The result implies that the remote tails of the resonance lines obey the same decay law as near the resonances. In fact, this statement generally does not take place since the relaxation mechanisms entering into the game at microwaves and the rf region of interest differ. Note also that only some mechanisms ("slow spin relaxation") are correctly described by the energy loss formulation (3). Besides, and this is rather essential, the bias field in our application, in contrast to the measurement of ΔH_k at microwaves, is not large enough to neglect losses due to magnetic hysteresis phenomena.

In view of these reasons the value of α can differ from the value of $g\Delta H_k/\omega_o$ significantly. Practically, we should rely on (5) with the α considered as an experimentally determined parameter, admitting its possible dependence on H_e and rf power level.

Remark 2. The field $\mathbf{H}$ in (5) is the field inside the garnets, it differs from the sum $\mathbf{H}_e + \mathbf{h}_e$. We denoted by the latter the external fields, outside of the garnet ring discs, at their surfaces. The components of $\mathbf{H}$ in the cylindrical coordinate system (r, φ, z) take the form

$$\mathbf{H} = (h_r,\ h_\varphi,\ H_e - 4\pi M_z) \tag{6}$$

where M_z is the axial component of $\mathbf{M}$; h_r and h_φ are the radial and azimutal components of $\mathbf{h}_e$. Such a form of $\mathbf{H}$ corresponds to the limit of small thickness of the ferrite discs as compared to other spatial scales. In particular, the rf component of $\mathbf{H}$, $\mathbf{h}$, differs from $\mathbf{h}_e$ by the axial demagnetizing field $(0, 0, -4\pi m_z)$ where m_z is the rf component of M_z. For the rf system in TRIUMF the component h_r is much less than h_φ and in estimates we adopt $h_r = 0$ for simplicity.

Linear regime.

In the linear rf regime, corresponding to the limit $\mathbf{h}_e \to 0$, the rf oscillations of $\mathbf{M}$, $\mathbf{m}(\mathbf{r}, t)$, have a small magnitude $m \ll M$. Neglecting the terms of order $\sim m^2/M^2$ the equation (5) reduces to the form

$$\dot{m}_r + \alpha \dot{m}_\varphi = -gH_o m_\varphi + gMh_\varphi \tag{7}$$

$$\dot{m}_\varphi - \alpha \dot{m}_r = gH_o m_r - gMh_r \tag{8}$$

where

$$H_o = H_e - 4\pi M.$$

The stationary solution of (7) and (8) for $h_r = 0$ and $h_\varphi \sim e^{i\omega t}$ reads, in terms of complex amplitudes of frequency ω,

$$m_\varphi = \frac{(\omega_o + i\alpha\omega)\omega_m}{(\omega_o + i\alpha\omega)^2 - \omega^2} h_\varphi \tag{9}$$

$$m_r = \frac{i\omega}{\omega_o + i\alpha\omega} m_\varphi \tag{10}$$

where $\omega_m = gM$ and

$$\omega_o = gH_o = g(H_e - 4\pi M)$$

is the natural frequency of the ferromagnetic resonance. For $\omega/2\pi = 50$ MHz and $H_o = 400$ Oe

$$\omega/\omega_o \approx 1/40.$$

The ratio $|m_r/m_\varphi|$, as seen from (9) and (10), is of the same small magnitude.

Approximately, for qualitive analysis and for estimates within the accuracy $\sim O(\omega^2/\omega_o^2)$

$$m_\varphi(\mathbf{r},t) = \frac{M}{H_o} h_\varphi(\mathbf{r}, t-\tau) \tag{11}$$

and

$$m_r(\mathbf{r},t) = \frac{M}{H_o\omega_o} \frac{\partial h_\varphi(\mathbf{r},t)}{\partial t}. \tag{12}$$

The field $H_o = H_e - 4\pi M$, being dependent on H_e, depends smoothly on t, with cycling frequency, and on $\mathbf{r}$. The delay time equals

$$\tau = \alpha/\omega_o$$

and is relatively small, $\omega\tau \ll 1$. For the considered case $\omega\tau \sim 10^{-4}$ and smaller.

The permeability component $\mu = \mu' - i\mu''$ determined as the ratio of b_φ/h_φ, where $b_\varphi = h_\varphi + 4\pi m_\varphi$ with m_φ given by (11), takes the form

$$\mu' = \frac{H_e}{H_o}, \quad \mu'' = \alpha\frac{\omega}{\omega_o}\frac{4\pi M}{H_o}$$

and the coresponding magnetic quality factor of the uniformly magnetized material is given by

$$Q_m = \frac{\mu'}{\mu''} = \frac{\omega_o}{\alpha\omega}\frac{H_e}{4\pi M}. \tag{13}$$

The quantity μ' is nothing but the static transverse permeability.

Magnetic loss versus ω and μ'.

The magnetic Q-factor Q_m as a function of μ' reads

$$Q_m = \frac{\mu'}{(\mu'-1)^2}\frac{g4\pi M}{\alpha\omega}. \tag{14}$$

The same formula for the magnetic Q was cited in [9] in a different approach. With decreasing of the operating frequency ω the factor Q_m increases like $1/\omega$. Such a feature is due to the viscous character of the magnetic damping in (5) which is proportional to the derivative of M [1]. Since the viscous damping disappears in the limit $\omega \to 0$, earlier or later a different kind of magnetic loss (e.g. associated with hysteresis phenomena) should become a factor. We may phenomenologically incorporate such a magnetic loss, including in α a frequency dependent contribution that increases as ω decreases. The smaller the ratio H_o/H_c, where H_c is the coercitive field, the larger is the magnetic loss of the hysteresis type. What is the contribution of the hysteresis mechanism of loss for the systems under consideration?

As evident from (14) the closer μ' is to 1, i.e. the larger the field H_o, the larger is the Q_m. However, the closer μ' is to 1 the harder it is to provide broad frequency band cavity tuning. Therefore the regimes with ultimately small H_o are preferable. To what extent it is possible to decrease H_o? Looking at (13) or (14) one may think that with decrease of H_o the Q_m decreases roughly linearly in H_o. But this is not so, since the value of α changes drastically with H_o due to the hysteresis magnetic loss mentioned above. Let us investigate this trend using the measurements performed by Smythe [10].

Fig. 2 shows the behaviour of Q_m and μ' versus $w/2\pi$ on the base of data [10a] for the ferrite garnet used in TRIUMF. The trend of α estimated from the data as

$$\alpha = \frac{\mu'}{(\mu'-1)^2}\frac{g4\pi M}{Q_m\omega} \tag{15}$$

is shown in Fig. 3. As seen the value of α is much below the figure estimated in sec. 2 from the microwave data of ΔH_k. The change of α with μ' in the working area of μ' makes it evident that the hysteresis type of magnetic loss is predominant. The same is found for all the types of magnetic garnets that are of interest for the rf tuning in which the Q_m and μ' were measured in [10b]. The corresponding estimates of α via the formula (15) are plotted in Fig. 4.

Thus, for the application under consideration the phenomenological parameter α determining the original model (5) cannot be regarded constant and its values differ by five and more times from that one expects from the data of ΔH_k at microwave frequencies.

Nonlinear regime.

Let us consider the nonlinear regime of moderate amplitudes of the rf oscillation of **m**, when the parameter s is small

$$s = \frac{M - \overline{M_z}}{M} \approx \frac{\overline{m_r^2} + \overline{m_\varphi^2}}{2M^2} \ll 1$$

but finite. The bar denotes averaging over period $2\pi/\omega$. The assumption $s \ll 1$ is relevant for the operating conditions. The regime $s \sim 1$ corresponds to the magnitude of rf field of order of H_o, so it corresponds to the hysteresis area of magnetization curve and large magnetic loss arises.

Using the asymptotic method of nonlinear mechanics [11] and neglecting the terms of the order of magnitude $\sim O(s^2)$ and $\sim O(\omega^2\tau^2)$, we obtain from (5)

$$\dot{m}_r + \alpha\dot{m}_\varphi = -(\omega_o + 4\pi\omega_m s_\varphi)m_\varphi + \omega_m(1 - s_\varphi)h_\varphi \tag{16}$$

[1]Note that a different trend, with μ'' and Q_m independent of ω, expounded in [8] on the basis of the same model (5), is a misunderstanding.

$$\dot{m}_\varphi - \alpha\dot{m}_r = (\omega_o + 4\pi\omega_m s_r)m_r - \omega_m(1 - s_r)h_r \tag{17}$$

where

$$s_\varphi = \frac{\overline{m_r^2} + 3\overline{m_\varphi^2}}{4M^2}$$

and s_r is given by the same expression with permutated m_r and m_φ. Note that the parameter α in these equations is a function of s.

The stationary regime of rf oscillations following from (16) and (17) is described by the formulae similar to that given by (9) and (10) but with coefficients depending on $s_{r,\varphi}$. In particular, for $h_r = 0$ we come to the approximate result similar to (11):

$$m_\varphi(\mathbf{r}, t) = \left(\frac{M}{H_o} - s_\varphi\frac{H_e M}{H_o^2}\right) h_e(\mathbf{r}, t - \tau). \tag{18}$$

Note that the approximation holds for s_φ small compared to 1 and in (18) the term $s_\varphi H_e M/H_o^2$ is always smaller than M/H_o. Note also that the parameter s_φ is a function of $\mathbf{r}$ and of smooth t. The field $h_e(\mathbf{r}, t)$ is to be determined selfconsistently, using Maxwell's equations in the cavity and the derived material relations.

Now the permeability $\mu' - i\mu''$, previously defined on page 5 as b_φ/h_φ and now determined from (18), is a function of the rf field intensity. Both μ' and μ'' decrease with growth of the intensity, but μ'' decreases faster which results in the ratio μ'/μ'' behaving as

$$Q_m(s) \approx (1 + as)Q_m \tag{19}$$

where Q_m is given by (13) in the limit $s \to 0$ and the factor a equals

$$a = \frac{3}{2} - \frac{1}{\alpha}\frac{\partial\alpha}{\partial s}$$

since $s_\varphi \approx 3s/2$. It view of (18)

$$s \approx \frac{1}{2}\frac{\overline{h_e^2}}{H_o^2}.$$

For the TRIUMF system at rf voltage $\sim$ 60 kV at the accelerating gap and $\mu' \sim 3$

$$\frac{\overline{h_e^2}}{H_o^2} \sim 5 \cdot 10^{-4}.$$

A considerable contribution to $\partial\alpha/\partial s$ is due to that α is a sharp function of H_o and the mean over $2\pi/\omega$ value $\overline{H}_o$ depends on s as $\overline{H}_o = H_e - 4\pi M(1 - s)$. Neglecting the other contributions results in

$$a = \frac{3}{2} + \frac{\mu'(\mu' - 1)}{\alpha}\frac{\partial\alpha}{\partial\mu'}.$$

This quantity is positive. Its estimate from the data plotted in Fig.3 gives at $\mu' = 3$

$$a \approx 6.$$

Note that the considered effect provides *increase* of the Q_m, (19), with growth of the rf power density in ferrite. Such a trend does not agree with the observations presented in [2] (Fig. 5 in [2]). Analysis of the discrepancy is out of the scope of the present note.

4. RF magnetic loss and Q-factor of the cavity

The energy of magnetic field absorbed in a unit volume of magnetic material per cycle of change of $\mathbf{H}$ is given by

$$\Delta u_m = -\frac{1}{4\pi}\oint \mathbf{B}d\mathbf{H}$$

where $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ is the magnetic induction. It follows for the rf power P_m absorbed in the garnets due to magnetic loss

$$P_m = -\frac{1}{4\pi}\int_{\Delta V} \overline{\mathbf{B}\dot{\mathbf{h}}}dV = \int_{\Delta V} \overline{\mathbf{h}\dot{\mathbf{m}}}dV \tag{20}$$

where ΔV is the volume filled by the garnets.

E.g. for $h_r = 0$, using (18), we obtain from (20)

$$P_m = \int_{\Delta V} \overline{h_\varphi \dot{m}_\varphi}dV = \alpha\frac{\omega^2 M}{\omega_o}\int_{\Delta V}\frac{\overline{h_\varphi^2}}{H_o}dV + \Delta P_m \tag{21}$$

where ΔP_m is a small nonlinear correction associated with the term $\sim s_\varphi$ in (18). Let us relate P_m to the quality factor of the cavity Q_{res}, determined by

$$\frac{1}{Q_{res}} = \frac{P_m}{\omega W} + \frac{P_o}{\omega W} = \frac{1}{Q_{res}^{(m)}} + \frac{1}{Q_{res}^{(o)}}$$

where P_o is all the other rf loss, W the total rf energy in the cavity, $Q_{res}^{(m)}$ the quality factor of the resonator with the loss given only by P_m, and $Q_{res}^{(o)}$ the quality factor of the cavity neglecting the magnetic loss. The energy W is twice the magnetic rf energy,

$$W = \frac{1}{4\pi}\int_V \overline{\mathbf{b}\mathbf{h}}dV$$

with V the volume of the cavity. In view of (21) we obtain, neglecting small nonlinear correction ΔP_m,

$$Q_{res}^{(m)} = \left(1 + \frac{H_o}{4\pi M k}\right)\frac{\omega_o}{\alpha\omega}. \tag{22}$$

In (22) and further H_o denotes a mean field, k is a dimensionless coefficient of filling of the resonator. More precisely, the H_o/k in (22) means the quantity

$$\frac{\int_V \overline{h_\varphi^2}dV}{\int_{\Delta V}(\overline{h_\varphi^2}/H_o)dV}.$$

Roughly

$$k = \frac{\int_{\Delta V}\overline{h_\varphi^2}dV}{\int_V \overline{h_\varphi^2}dV} \sim \frac{\Delta V}{V}.$$

It follows from (22) and (13)

$$Q_{res}^{(m)} = \left(1 + \frac{1-k}{k\mu'}\right)Q_m \tag{23}$$

where now Q_m is a mean of (13). For $k = 1$ the $Q_{res}^{(m)}$ reduces to Q_m.

For the conditions

$$\alpha = 2\cdot 10^{-3}, \quad \frac{\omega}{2\pi} = 50\ \mathrm{MHz}, \quad \mu' = 3 \quad \text{and} \quad k = 0.5$$

the eqs. (14),(23) result in

$$Q_m \approx 1.5\cdot 10^4, \quad Q_{res}^{(m)} \approx 2\cdot 10^4 .$$

The nonlinear correction ΔP_m in (21) is of order $\sim s_\varphi H_e/H_o$ and is of negative sign, as is evident from the form of (18). It should be again emphasized that such a trend corresponds to the conditions of α constant, independent on s, which is not the case for the systems of interest. Roughly, the effect is in that the Q_m in (23) should be replaced by the $Q_m(s)$ given by (19).

The derived relations between $Q_{res}, Q_{res}^{(m)}$ and the magnetic Q factor of ferrite garnet allow one to estimate the parameter α and determine its nature by investigating the trend of α as a function of H_o and s from the rf system behaviour.

5. Nonlinear shift of resonance frequency

Let us investigate the trend of the nonlinear shift $\delta\omega_{res}$ caused by the fast nonlinear process in garnet on the natural frequency ω_{res} of the rf cavity as a function of rf power. A variety of slow nonlinear processes, first of all thermal and thermo-elastic ones, is not taken into account in this section.

Simple approximate formulae can be derived by disregarding spatial inhomogeneity of the magnetic permeability μ'. Then

$$\delta\omega_{res} = \frac{\partial\omega_{res}}{\partial\mu'}\delta\mu'$$

where $\delta\mu'$ is the increment of μ' caused by increase of the rf power density. In view of (18)

$$\delta\mu' = -\frac{4\pi M H_e}{H_o^2}s_\varphi = -\mu'(\mu'-1)s_\varphi$$

and

$$s_\varphi \approx \frac{3}{4}\frac{\overline{h_e^2}}{H_o^2}.$$

So, approximately

$$\frac{\delta\omega_{res}}{\omega_{res}} = \beta\frac{\overline{h_e^2}}{H_o^2} \tag{24}$$

with

$$\beta = -\frac{3}{4}(\mu'-1)\frac{\mu'}{\omega_{res}}\frac{\partial\omega_{res}}{\partial\mu'}. \tag{25}$$

The convenience of these formulae for analysis of the nonlinear frequency shift is in that one can readily estimate the parameter β by measuring ω_{res} as a function of μ' at *low* rf powers. The derivative $\partial\omega_{res}/\partial\mu'$ is of negative sign, so the β is positive, i.e. the resonance frequency shifts toward positive values of ω with growth of rf power.

For the TRIUMF system, using the data potted in Fig. 2, one obtains at $\mu' \sim 3$ the estimate

$$\beta \approx \frac{1}{2}. \tag{26}$$

Let us present another approach allowing to make a rated estimate of the $\delta\omega_{res}$. At resonance the magnetic and electric rf energy in the cavity coincide, i.e.

$$\frac{1}{8\pi}\int_V \epsilon\overline{\mathbf{e}^2}dV = \frac{1}{8\pi}\int_V \overline{\mathbf{bh}}dV \tag{27}$$

where $\mathbf{b}$ is the rf part of magnetic induction, $\epsilon = \epsilon(\mathbf{r})$ is the dielectric permeability, the electric rf field $\mathbf{e}$ relates to the magnetic field by the Maxwell's equation

$$rot\ \mathbf{h} = \frac{\epsilon}{c}\frac{\partial \mathbf{e}}{\partial t}$$

with c the light velocity. From these equations it follows for the resonance frequency ω_{res}

$$\omega_{res}^2 = \frac{\int_V \overline{(rot\ \mathbf{h})\,\epsilon^{-1} rot\ \mathbf{h}}dV}{c^2\int_V \overline{\mathbf{bh}}dV}. \tag{28}$$

For the geometry under consideration the spatial profile of $rot\mathbf{h}$ practically does not change with the growth of the rf power level. Therefore the change in ω_{res} is due to changes in the denominator of (27). So we come to the following formula for the nonlinear shift $\delta\omega_{res}$

$$\frac{\delta\omega_{res}}{\omega_{res}} = -\frac{1}{2}\frac{\int_{\Delta V}\overline{\mathbf{h}\delta\mathbf{b}}dV}{\int_V \overline{\mathbf{hb}}dV} \tag{29}$$

where $\delta\mathbf{b}$ is the difference between total $\mathbf{b}$ and its linear part. In view of (18) and since

$$s_\varphi \approx \frac{3}{4}\frac{\overline{h_e^2}}{H_o^2}$$

this results in

$$\frac{\delta\omega_{res}}{\omega_{res}} \approx \frac{3}{8}\left[\frac{1}{4\pi}\int_V \overline{h_e^2}dV + \int_{\Delta V}\frac{M}{H_o}\overline{h_e^2}dV\right]^{-1}\int_{\Delta V}\frac{H_e M}{H_o^2}\frac{\overline{h_e^2}^2}{H_o^2}dV. \tag{30}$$

Eq. (30) can be rewritten in the form

$$\frac{\delta\omega_{res}}{\omega_{res}} = \frac{3}{8}\left(1+\frac{H_o}{4\pi Mk}\right)^{-1}\frac{H_e}{H_o}\frac{\overline{h_e^2}}{H_o^2} \tag{31}$$

where now H_o, H_e and $\overline{h_e^2}$ denote some spatial averages of the corresponding local fields.

Comparing (31) with (24), we thus obtain a rated estimate of β. In terms of μ' (which is a spatial average now) it reads

$$\beta = \frac{3}{8}\frac{\mu'(\mu'-1)}{\mu'-1+\frac{1}{k}}. \tag{32}$$

Taking $\mu' \sim 3$ and $k \sim 1/2$ one arrives at the estimate (26). So both approaches are in a good agreement. As follows from (32), since $k \leq 1$, the value of β cannot be larger than

$$\beta_{max} = \frac{3}{8}(\mu'-1).$$

6. Static instability

There are several kinds of instabilities caused by ferrite garnet that may appear with increasing of the rf power level. Most obvious is the "static" instability: the nonlinear frequency shift results in such a deformation of the resonance curve that the curve takes a hysteresis form and corresponding jumps of the stationary regime arise. Let us estimate the threshold of static instability caused by the fast nonlinear process.

We will use the notation

$$y = \frac{\overline{h_e^2}}{H_o^2}.$$

The resonance curve is given by the equation

$$\frac{y}{y_{max}} = \frac{1}{Q_{res}^2 \xi^2(y) + 1} \tag{33}$$

where y_{max} denotes the value of y at the resonance $\omega = \omega_{res}$ and

$$\xi = \frac{\omega}{\omega_{res}} - \frac{\omega_{res}}{\omega}$$

is a dimensionless resonance detuning. According to sec. 5 the dependence of ξ on y may be approximated near the resonance by the formula

$$\xi(y) = \xi_o - 2\beta y \tag{34}$$

where $\xi_o = \xi(0)$ and the coefficient β is given by the expressions (25),(32). The magnitude of β estimated both from the calculation and experimental data is given by (26). In view of the dependence in (19) the Q_{res} in (33) is a function of y as well and the magnitude of the derivative $\partial Q_{res}/\partial y$ exceeds that of $\partial\xi/\partial y$ by the factor $a/2\beta \sim 6$. However, this dependence is not essential for the static instability and can be neglected in the analysis. As a result, we arrive at a cubic algebraic equation for determining y as a function of the working frequency ω. The points $\xi_\pm$ of detuning ξ_o where the hysteresis jumps occur are given by

$$\xi_\pm = 4\beta y \pm \sqrt{4\beta^2 y^2 - Q_{res}^{-2}}.$$

Denoting by y_{hys} the minimum value of y at which the hysteresis dependence in $y(\omega)$ appears, we obtain the criterion

$$y_{hys} = \frac{1}{2\beta Q_{res}}. \tag{35}$$

For the cavity in TRIUMF the Q_{res} varies with change of H_e (due to change of Q_m) and is about $3 \cdot 10^3$ at $\mu' = 3$, so in view of the estimate (26)

$$y_{hys} \sim 3 \cdot 10^{-4}$$

for the case. This value is less than the estimate $5 \cdot 10^{-4}$ of y cited in sec. 3 for the operating regime of 60 kV at the accelerating gap. So the system reaches the threshold of instability.

7. Dynamic instabilities

Besides the static instability there should arise "dynamic" instabilities, appearing as an automodulation (labeled as modulation in Figs. 5,6) of the rf output. Some of them precede the static instability, i.e. their threshold $y_{aut} < y_{hys}$, as schematically shown in Fig. 5. The area "hysteresis jumps" corresponds to the interval where the intermediate and unstable branch of the resonance curve $y(\xi_o)$ given by (33),(34) appears. Slow magnetic relaxation resulting in the dependence $\alpha(s)$, thermal and striction processes should play an essential role in the automodulation instabilities.

Note that even in empty cavities with rather thick ($\sim$ 10 mm) copper walls, the mechanical forces exerted on the walls by the rf field are able to cause low frequency vibration appearing as an automodulation, as was observed and described for the first time in [12]. A general theory of the striction phenomena in the resonators, both empty and filled with magneto- or electro-elastic material, was developed in [13] for the case when the nonlinearity due to striction is predominant over the other nonlinearities of the rf system. Then the trend of striction instabilities is as shown in Fig. 6. The theory of the combined process, incorporating the cubic nonlinearity like the magnetic nonlinearity of permeability considered above, is similar to that developed in [14] for the magnetoelastic phenomena in ferromagnets in the vicinity of ferromagnetic resonances. Let us present a brief analysis.

Let $\mathbf{F}(\mathbf{r})$ denote a distribution of the forces averaged over time $2\pi/\omega$ acting on an element of the ferrite (or another element of the cavity) from the rf field. The forces cause a deformation distribution $\mathbf{d}(\mathbf{r})$. The latter changes the resonance frequency ω_{res}. From energy considerations it follows

$$\mathbf{F}(\mathbf{r}) = -\frac{W}{\omega_{res}} \frac{\delta\omega_{res}}{\delta\mathbf{d}(\mathbf{r})} \tag{36}$$

with $\delta/\delta\mathbf{d}(\mathbf{r})$ the functional derivative. The rf energy W of the cavity is a sharp function of the resonance detuning ξ_o and depends on it with a delay since the rf regime reacts to change of ξ_o with delay. Thus, the force (36) is a sharp and retarded functional of $\mathbf{d}(\mathbf{r},t)$ which produces the effective elasticity and friction. Their measure is given by the quantity

$$K = -\frac{\delta\mathbf{F}(\mathbf{r},t)}{\delta\mathbf{d}(\mathbf{r}',t')} = -\psi\mathrm{K} \tag{37}$$

where ψ is the dimensionless function

$$\psi(t,t') = \frac{1}{y}\frac{\delta y(t)}{\delta\xi_o(t')}$$

changing sign depending on the detuning ξ_o and

$$\mathrm{K} \approx \frac{2W}{\omega_{res}^2}\frac{\delta\omega_{res}}{\delta\mathbf{d}(\mathbf{r})}\frac{\delta\omega_{res}}{\delta\mathbf{d}(\mathbf{r}')}$$

represents a scale of non-negative elasticity produced by the rf forces for the system. In terms of spectral representation, for a harmonic in t vibration $\mathbf{d}(\mathbf{r},t) \sim e^{i\Omega t}$, the quantity K reduces to

$$K = K'(\Omega) - iK''(\Omega)$$

where K' and K'' represent effective elasticity and friction of the mean rf forces. In the limit $\Omega\tau_d \ll 1$, where τ_d is a time delay scale of the reaction of W to ξ_o (τ_d depends on ω_o/Q_{res}, y and ξ), these quantities take the form

$$K' = \psi_o \mathrm{K}, \qquad K'' = \Omega\tau_d\psi_o\mathrm{K} \tag{38}$$

with the K given above , the dimensionless function

$$\psi_o = -\frac{2Q^2\xi}{1+Q^2(\xi^2-4\beta y\xi)}$$

and $\xi = \xi_o - 2\beta y$.

The quantity K'' in (38) represents a viscous friction. If $\xi < 0$ then $\psi_o > 0$ and the effective friction is positive, producing additional damping of mechanical vibration. When $\xi > 0$ the friction is negative and tends to build up vibration in an area of plane (y, ξ_o) out of the hysteresis jumps. Note that the statement about the signs does not depend on the form of vibration. In the vicinity of vertical regions of the resonance curve $y(\xi_o)$ $\psi_o \to \infty$ indicating a huge increase of the negative friction. When it exceeds intrinsic mechanical friction for a mode of low frequency vibration in the system, there should arise automodulation. The corresponding area of instability is adjacent to the area of hysteresis jumps (Fig. 5). Note that the estimate of the effective negative friction given by (38) corresponds to the infinitesimal mechanical vibration. The threshold of instability towards finite values of the mechanical perturbations may precede it covering a larger, from the left side, area of the plane (y, ξ_o) than is plotted on Fig. 5.

Another important type of slow nonlinear processes is thermal. The main contribution to it is due to the fact that the magnitude $M = |\mathbf{M}|$ is a decreasing function of the temperature. So, M decreases with y and via $H_o = H_e - 4\pi M$ this leads to the μ' decreasing with y. Hence the quantity $\delta\xi/\delta T$ produced by the thermal process is negative, tending to deform the resonance curve as the considered magnetic nonlinearity. Thus the temperature distribution $T(\mathbf{r},t)$ contains a contribution sharply dependent on the resonance detuning and we may introduce, similarly to the quasilesticity of the mechanical vibration, the effective quasielasticity of the behaviour of $T(\mathbf{r},t)$ produced by the rf heating of the ferrite. The effective friction of this quasielasticity is negative at $\xi < 0$, while at $\xi > 0$ the effective rigidity is negative. In principle, both the thermal and striction process together can provide the thermoelastic oscillation on the right side of the resonance curve and on the left as well. Near the threshold of the thermal instability the frequency of automodulation should have a square-root dependence

$$\Omega \sim \sqrt{y - y_{aut}}$$

while the frequencies of the striction type of automodulation change insignificantly.

One more important mechanism of the automodulation is due to the relaxation dependence $\alpha(s)$ resulting in $Q_{res}(y)$ due to $Q_m(s)$ given by (19). The magnetic relaxation process involved is faster than the thermal but still is a slow one, in our classification. The characteristic time scale T_o of the response of α to s is larger than $2\pi/\omega$. Since the process is less inertial than the thermal one, its role for the automodulation is expected to be more essential. We have not found any literature on the dynamics of α. Infinitesimal oscillation $\delta\alpha(t)$ in response to variation of the rf field regime near its stationary state may be approximated by the equation

$$T_o \frac{\partial \delta\alpha}{\partial t} + \delta\alpha = \frac{\partial \alpha}{\partial y} \delta y. \tag{39}$$

The contribution to $\partial\alpha/\partial y$ neglecting the thermal process, due to change of the mean field $\overline{H}_o$, was estimated in the end of sec. 3; it is negative and of magnitude $\mu'(\mu'-1)\partial\alpha/\partial\mu' \sim 5\alpha$. The variation δy in (39) is a retarded functional of $\delta\alpha$. Roughly,

$$\delta y(t) = \frac{\partial y}{\partial Q_{res}} \frac{\partial Q_{res}}{\partial \alpha} \delta\alpha(t - \tau_d).$$

Here τ_d is the same as in (38). The derivatives in the right can be estimated from (33),(34),(22) and the relation between Q_{res} and $Q_{res}^{(m)}$ given in sec. 4. As a result, provided the automodulation frequency $\Omega \ll 1/\tau_d$, the instability area is determined by the inequality

$$\frac{2\xi^2 y}{(\xi_o - \xi_+)(\xi_o - \xi_-)} \frac{\tau_d Q_{res}}{T_o Q_{res}^{(m)}} \frac{\mu'(\mu'-1)}{\alpha} \frac{\partial\alpha}{\partial\mu'} \geq 1. \tag{40}$$

Note that $\xi_\pm$ and τ_d are functions of y. It follows from (40) that the instability should occur on both sides of the resonance curve. On the side $\xi > 0$ its area is adjacent to the area "hysteresis jumps", overlaping with the area "striction modulation". On the side $\xi < 0$ it is located at larger values of y. In Fig. 5 it is labeled as thermomagnetic modulation, since in fact the thermal process may be involved and give a considerable change of the trend. The area is shown schematically, not to scale. Similar to the thermal automodulation the frequency of the automodulation should have a square-root dependency on y. This can be used to distinguish mechanisms.

The effects become important when the increments of instabilities become comparable with and exceed the synchrotron cycling frequency. Then they restrict operation of the system, unless special measures are taken, like design of feedback loops to suppress the automodulation process.

Throughout we assumed the rf oscillation of $\mathbf{M}(\mathbf{r}, t)$ in the ferrite garnet being spatially smooth. In principle, the regime can become unstable towards parametric excitation of short-scale magnetic oscillation at frequencies $\sim \omega/2$ or $\sim \omega$; generation at multiple frequencies of ω are less plausible. The instabilities could significantly decrease the effective magnetic Q, imposing serious limits on the operation of the system. Note that the position of spectrum and interactions essential for the generation depend on the inhomogeneities in the material and the bias field. In fact, the more the biasing magnetic field exceeds the hysteresis area the higher the frequencies of the dangerous resonances and the higher are the instability thresholds.

However, in view of magnetoelastic interactions in ferrite garnet there exist modes of vibration at frequencies $\omega/2$ and ω that cannot be removed and might be, in principle, dangerous because of their possible parametrical excitation at high rf power levels. Again, the effect should critically depend on the density of inhomogenities of the spatial scales $l \sim v/\omega$, where v is the velocity of elastic waves in the material; roughly $l \sim 10^{-2}$ mm.

Any observation or mentioning of these many instabilities in the rf cavities with transversely biased ferrite garnet is unknown to us. So we stop the discussion at the sketchy stage presented above.

8. Conclusion and Recommendations

Proceeding from a general model of magnetic behaviour, given by (5) with a phenomenological parameter α determining the magnetic loss, the investigation showed that for the conditions of interest this parameter is a sharp function of the bias field rather than a constant. The values that α takes differ by five and more times from the values conventionally estimated via the linewidths of the spin resonances at microwaves. These points should be taken into consideration when determining the trend of magnetic Q's and searching for optimal regimes and materials.

The physics of such a trend of α, since it manifestss even at low level rf fields, was associated with the hysteresis phenomena. In this connection systematic measurements of the frequency dependency of α would be of interest.

The consideration of magnetic loss as a function of rf power at levels below the instability thresholds, lead to the conclusion that both the dependence of α on the internal field and the nonlinearity of magnetic subsystem with $\alpha = 0$ should cause increase of magnetic Q with rf power. This is not in agreement with observations reported at Los Alamos [2]. Although possible explanations have been given, further detailed analysis and experimental measurements would be necessary to fully understand the discrepancy.

The nonlinear frequency shift of the rf resonant regime caused by the fast nonlinear process was investigated as a function of rf power. Two different methods of analysis were developed and simple formulae for the scale β specifying the nonlinearity and its dependence on the system's parameters and rf regime were derived. Both methods lead to the same estimate thus providing its reliability. This allows one to single out the nonlinear effect of slow processes and put their measurement on a clear quantitative basis.

As shown, the nonlinear effect caused by ferrite garnet at high rf power can give rise to a number of instabilities of the rf regime. First of all, it is the instability manifesting in hysteresis jumps of the rf regime due to the nonlinear frequency shift. According to the consideration this static instability should be preceded by dynamic instabilities displaying in low frequency modulation of the rf regime together with elastic and thermo-magnetic low frequency oscillation in the system. The TRIUMF system reaches the estimated threshold of the static instability and further detailed analysis should be of interest, especially if higher voltages are considered for the cavity.

The developed method of analysis of magnetic loss and nonlinear effect may find applications in various high power rf cavities with transversely biased ferrite garnet.

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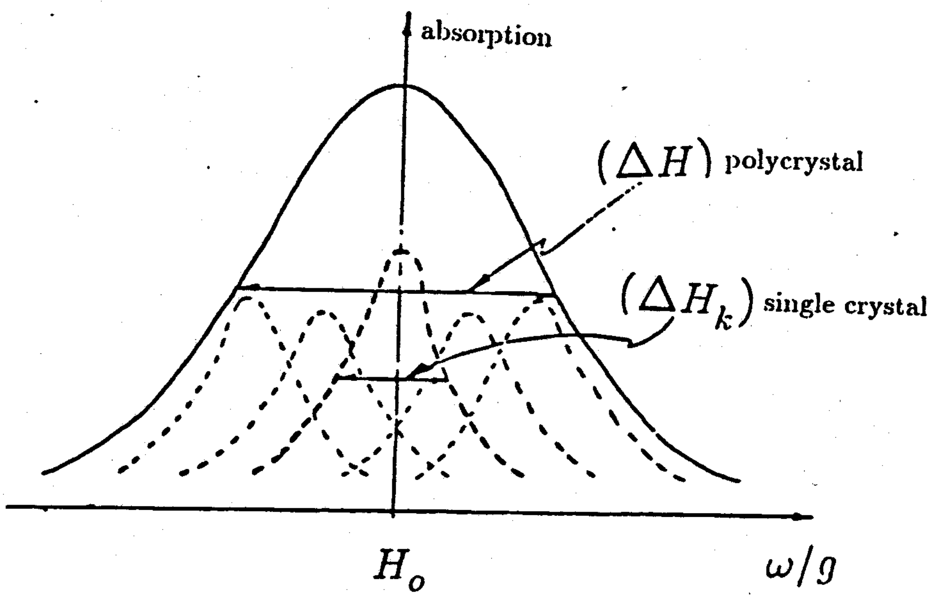

Fig. 1.

The resonance curve of the whole sample (solid line) is the sum of the dotted curves, representing the resonance curves of the subvolumes. Far from the resonances the tails of the curves are reduced to the same form.

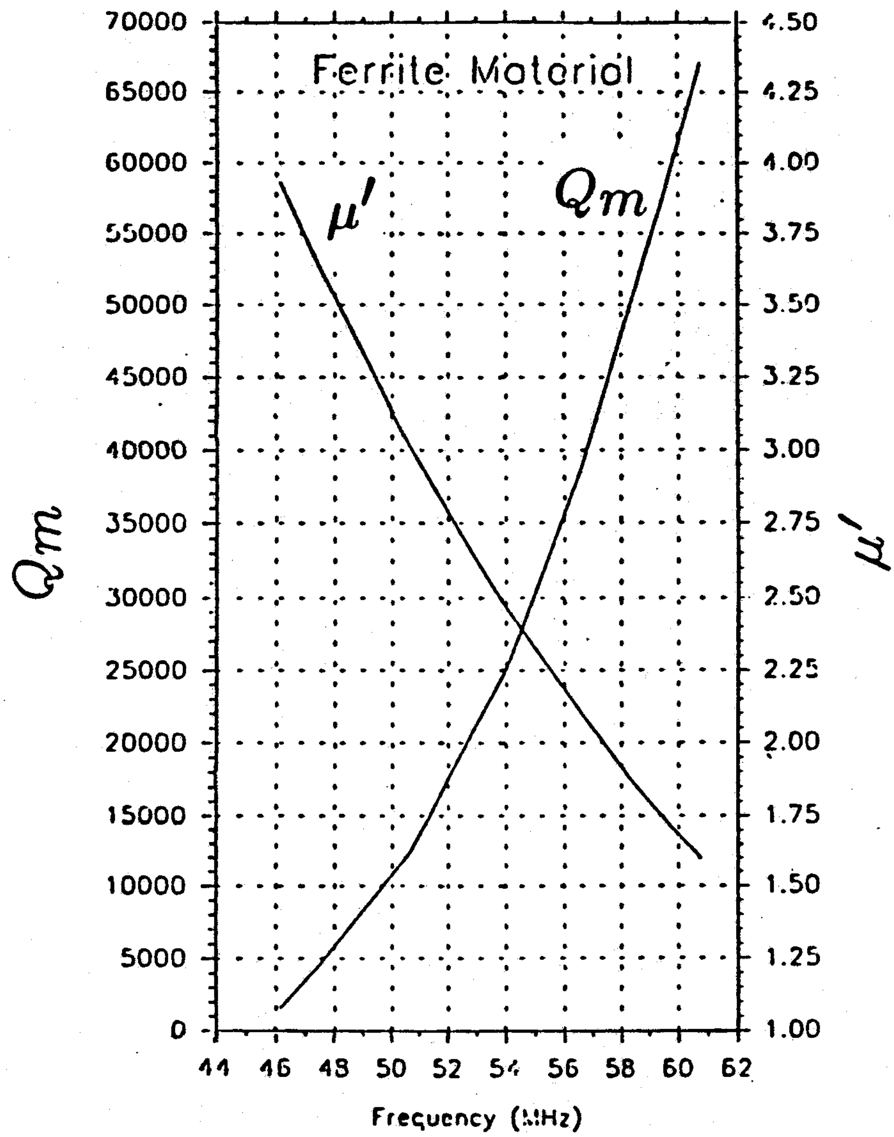

Fig. 2. Behaviour of Q_m and μ' for the ferrite garnet used in the TRIUMF system

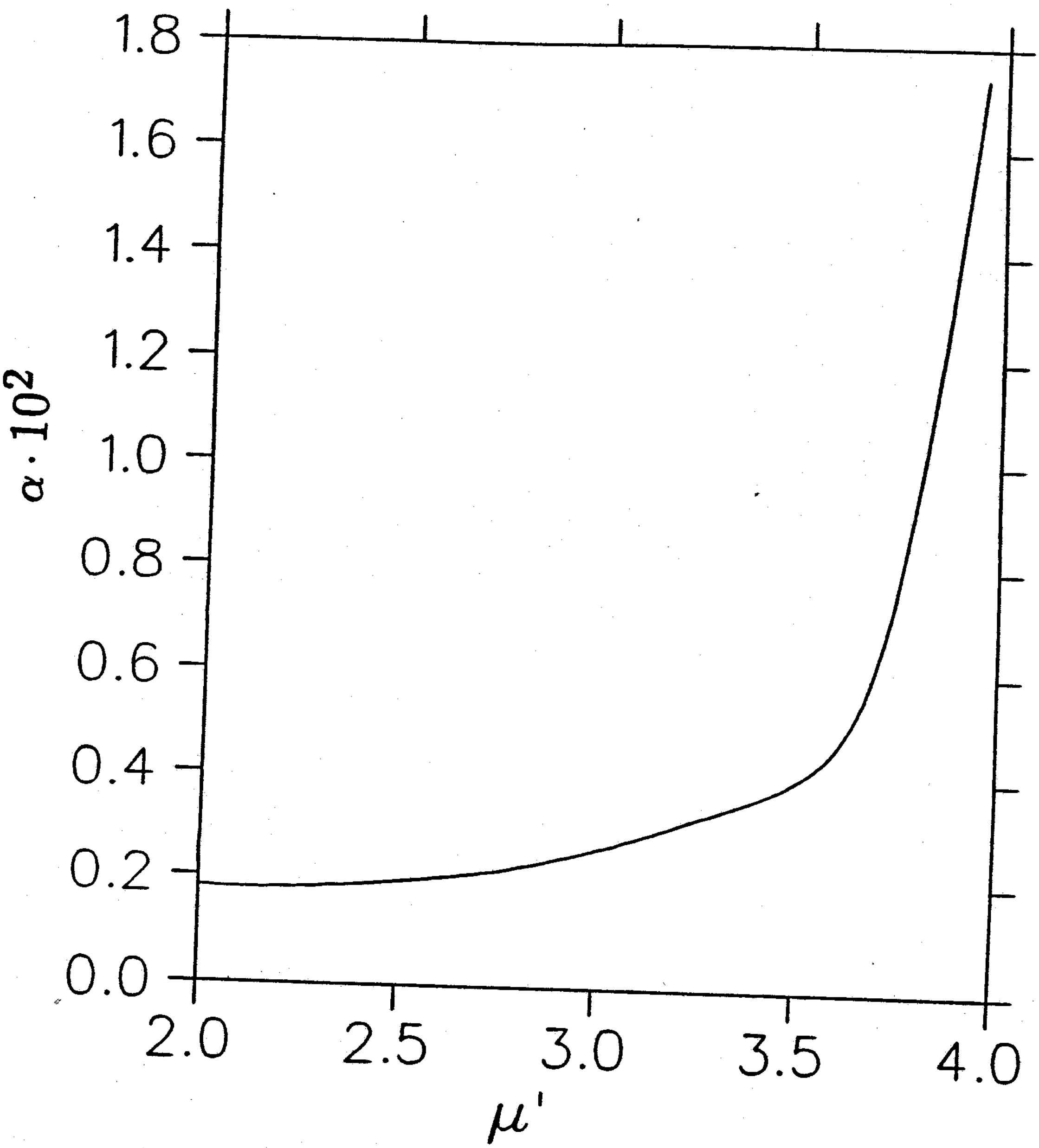

Fig. 3. Behaviour of α estimated by the formula (15)

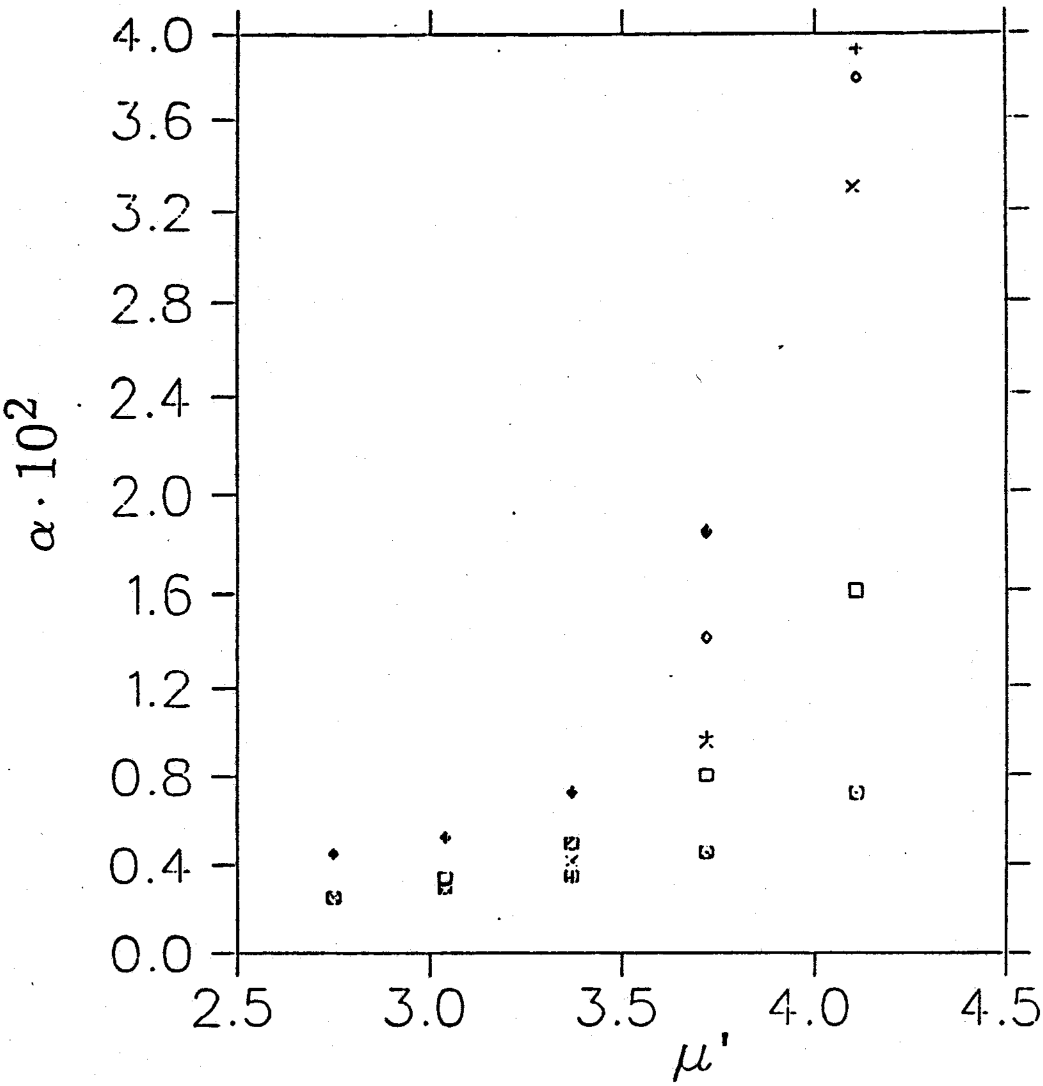

Fig. 4. α estimated by (15) for different types of ferrite garnets from the measurements presented in [10b]. In the working interval of interest α changes with μ' considerably for all types.

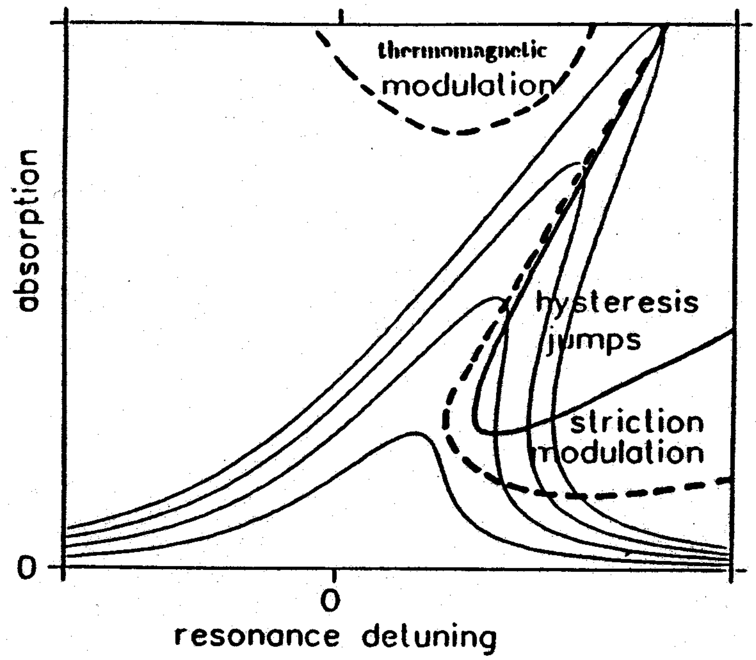

Fig. 5. Areas of instabilities.

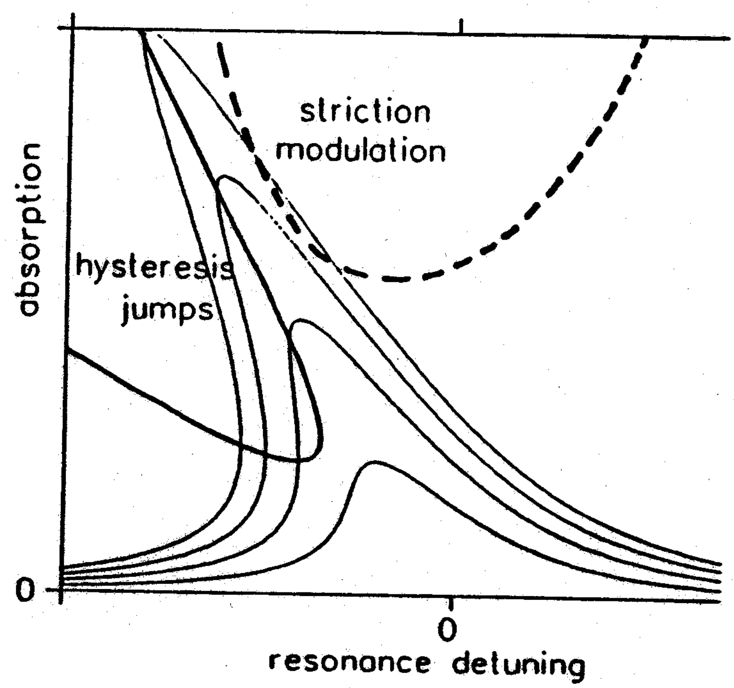

Fig. 6. Instabilities in empty cavity caused by the striction effect.